

Period integrals of automorphic forms

The purpose of the talk is to give a birds view of some aspects of period integrals and their relation with other topics of interest of automorphic forms and aut. reps.

Classical Example:

$$E(z, s) = \sum' \frac{\text{Im}(z)^s}{|mz+n|^{2s}} \quad \begin{array}{l} \text{Normalized real analytic} \\ \text{Eisenstein series on } \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \end{array}$$

$$E(\sqrt{4}, s) = \zeta(s) L(s, \chi_{-4}) \quad \begin{array}{l} \text{quad. char. mod 4} \\ \downarrow \\ = \zeta_{\mathbb{Q}(\sqrt{-4})}(s) \end{array}$$

WalDSPurger (86'): f Hecke-Maass cusp form on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$

$$\|f\|^{-2} |f(\sqrt{4})|^2 \approx L(\frac{1}{2}, f) L(\frac{1}{2}, f \otimes \chi_{-4}) = L(\frac{1}{2}, bc(f))$$

$bc: \text{Aut}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) \longrightarrow \text{Aut}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_3)$ In this context of complex extension 1st defined by Shimura (79') and generalized by Langlands (80') and Arthur-Clozel (89')

Both formulas generalize to weighted sums of Heegner points of a fixed negative discriminant (replacing -4).

Adelic version:

$$\begin{aligned}
 \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} &\simeq \mathbb{Z}(\mathbb{A}) \backslash \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / \mathcal{O}(\mathbb{Z}) \times \prod_p \mathrm{GL}_2(\mathbb{Z}_p) \\
 f\text{-modular form} &\rightsquigarrow \phi_f \text{- aut. form on } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \rightsquigarrow \pi_f \text{- aut. rep. of } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})
 \end{aligned}$$

\exists torus $T \subseteq \mathrm{GL}_2$, $T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\}$ s.t. $T(\mathbb{Q}) \simeq \mathbb{Q}[\sqrt{-1}]^\times$

$$f(\sqrt{-1}) = \int_{\mathbb{Z}(\mathbb{A}) \backslash T(\mathbb{Q}) \backslash T(\mathbb{A}_{\mathbb{Q}})} \phi_f(t) dt = \mathcal{P}_T(\phi_f) \quad \text{Period integral of } \phi_f$$

General Framework

F - # field, $\mathbb{A} = \mathbb{A}_F$

G - reductive gp. / F

H - closed subgp.

$\phi: G(\mathbb{A}) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}$ cont. aut. function

$$\mathcal{P}_{H, \chi}(\phi) = \int_{H(\mathbb{A}) \backslash H(\mathbb{A}_{\mathbb{Q}})} \chi(h) \phi(h) dh$$

Rule: H -semi-simple, ϕ -cusp. \mathcal{P}_H is abs. convergent (Ash-Ginzburg-Rallis)

Def: An aut. rep. π of $G(\mathbb{A})$ is H -distinguished if $\mathcal{P}_H|_{\pi} \neq 0$.

(Whittaker functional = $\mathcal{P}_{N, \psi}$; N -max't unip. ψ -non-deg. on N)

We already saw a relation of Period integrals to special L -values.

Often, distinction also characterizes the image of a functorial lifting. Periods are then related to Langlands functoriality.

Example: Unitary Periods & quad. base-change

E/F - quad. ext. of # fields.

$\zeta = \zeta_{E/F}$ the quad. char. of $N(A_{E/F}) \times A_F^\times$.

$$\begin{array}{ccc} \text{bc} : \text{Aut. Reps } GL_n(A_F) & \longrightarrow & \text{Aut. Reps } GL_n(A_E) \\ \pi' & \longmapsto & \pi = \text{bc}(\pi') \end{array}$$

$$L(s, \pi) = L(s, \pi') L(s, \pi' \otimes \zeta)$$

Characterization of the image: π -irr. cusp. aut. rep. of $GL_n(A_E)$

$$\begin{array}{ccc} \pi \text{ is Galois invariant} & \iff & \pi = \text{bc}(\pi') \text{ is} \\ (\text{w.r.t. } E/F) & \uparrow & \text{a q. base-change} \\ \text{Arthur-Cloze (1991)} & & \text{Jacquet (2010)} \end{array} \iff \begin{array}{l} \pi \text{ is } H\text{-dist. where} \\ H = U_n(E/F) \text{ - q. split unitary gp.} \end{array}$$

Relation with L-values: $\pi = \text{bc}(\pi') \ni \phi$

$$\|\phi\|_2^2 \left| \mathcal{P}_{U_n(E/F)}(\phi) \right|^2 \approx \frac{L(1, \pi' \otimes \bar{\pi}' \otimes \zeta)}{L(1, \pi', Ad)}$$

Unlike Waldspurger's formula here the L-values are at the edge of the critical strip and we know more about them.

When considering compact unitary periods they become sums of points evaluations. Thus known lower bounds to the RHS indicate that aut. forms that are base-change lifts have

large L^∞ -norms. This is compatible with a conjecture of Sarnak on L^∞ -norms of aut. forms.

To obtain these results Jacquet developed, over many years, his Relative Trace Formula. At the heart of the relation between Unitary Periods and base-change is a geometric relation

between double cosets.

$$N_n(F) \backslash GL_n(F) / N_n(F), \psi' \approx N_n(E) \backslash GL_n(E) / U_n(E/F)$$

Another Setting: Orthogonal Periods $(G, H) = (GL_n, O_n)$

$$N_n \backslash \widetilde{GL}_n / N_n, \psi \approx N_n \backslash GL_n / O_n$$

$$\text{Aut. Repr.}(\widetilde{GL}_n) \xrightarrow[\text{(Shimura cor.)}]{\text{Flicker-Kazhdan}} \text{Aut. Repr.}(GL_n)$$

Jacquet's conjecture: Orthogonal Periods on GL_n are

related to Whittaker coeffs on \widetilde{GL}_n via F-K.

Hence by Brubaker-Bump-Friedberg to A_n , WMOS.

Evidence for $GL_3(\mathbb{Z})$ Eisenstein series: Chinta-Offen,
Li-Mei Lin

Recently: Viet-Cuong proved the Fundamental Lemma for the relevant RTF.

Given the conjecture one could hope to understand orth. period of Eis. series via WMOS. But for cusp forms maybe we could

relate orth. periods to metaplectic Whittaker, but we understand neither.

Def: (G, H) is a vanishing pair if

NO cusp. rep. of G is H -dist.

Example: (G, G) is a vanishing pair since $P_G(\mathfrak{d}) = \langle \mathfrak{d}, 1 \rangle, \perp L^2_{\text{cusp}}$.

The following are vanishing pairs:

Jacquet-Rallis	(GL_{2n}, Sp_n)		
A-G-R	}	$(GL_{n+m}, GL_n \times GL_m)$	$n \neq m$
		$(Sp_{n+m}, Sp_n \times Sp_m)$	
		}	(GL_{n+m}, GL_m) $n > m$ Friedberg - J

What about the non-cuspidal spectrum?

$$L^2(G(\mathbb{F}) \backslash G(\mathbb{A})) = L^2_{\text{disc}}(G) \oplus L^2_{\text{cont}}(G)$$

$$\parallel$$

$$L^2_{\text{cusp}}(G) \oplus L^2_{\text{res}}(G)$$

Example: $L^2_{\text{disc}}(GL_n) = \bigoplus U(\sigma, k)$

$n = km, \sigma$ -irr. cusp. out. rep. of $GL_m(\mathbb{A})$

$$\pi U(\sigma, k) = L\mathbb{Q} \text{ Ind}(\sigma | 1 | \dots | \sigma | 1 | \dots)$$

Thm (Symplectic Periods) $(G, H) = (GL_n, Sp_n)$

① $P_H \upharpoonright U(\sigma, k)$ is abs. conv.

② $U(\sigma, k)$ is H -dist. iff k is even

$$L^2_{\text{cont.}}(GL_m) = \bigoplus_{(L, \Gamma) \in \mathbb{R}^m} \int \text{Ind}(U(\sigma_1, k_1) | \cdot|^{\lambda_1} \otimes \dots \otimes U(\sigma_m, k_m) | \cdot|^{\lambda_m}) d\lambda$$

discrete data on L proper Levi
 $\Gamma = \bigoplus_i U(\sigma_i, k_i)$

↑ associated to
Eis. series
 $E(\mathfrak{g}, \Gamma, \lambda)$

Thm (Yamagata): $(G, H) = (GL_m, Sp_n)$

① $P_H(E(\Gamma, \lambda))$ is abs. conv. for $\lambda \in i\mathbb{R}^m$

② $P_H |_{\text{Ind}(\Gamma, \lambda)} \neq 0$ iff $\Gamma = \bigoplus_i U(\sigma_i, k_i)$

The case $(G, H) = (Sp_{2n}, Sp_n \times Sp_n)$

H -distinction is related to the descent construction of

Ginzburg-Rallis-Soudry. $\text{Aut}(Sp_{2n}) \xrightarrow{\text{FJ}} \text{Aut}(\tilde{Sp}_n)$

Let $\tau = \tau_1 \otimes \dots \otimes \tau_k$ irr. cusp. ant. rep. of $GL_{2n_1} \times \dots \times GL_{2n_k}$, s.t.

τ_i is $(GL_{n_i} \times GL_{n_i})$ -dist. $\begin{matrix} \text{(Baruch-Friedberg)} \\ \Leftrightarrow \\ \text{(F-J)} \end{matrix}$ $L(\frac{1}{2}, \tau_i) \neq 0, L(1, \tau_i, \Lambda^2) = \infty$

$\tau_i \neq \tau_j \forall i \neq j$.

Assume $n = n_1 + \dots + n_k$.

Let M be the Levi of $Sp_{2n} \simeq GL_{2n_1} \times \dots \times GL_{2n_k}$

consider τ as a cusp. rep. of M .

$$\begin{array}{ccc} \text{Ind}^{Sp_n}(\tau_1 | \cdot|^{\lambda_1} \otimes \dots \otimes \tau_k | \cdot|^{\lambda_k}) & \xrightarrow{\lambda = (\frac{1}{2}, \dots, \frac{1}{2})} & \mathcal{E}_{-1}^\tau - \text{irr. rep.} \\ \downarrow & \searrow \text{Res}_{\lambda = (\frac{1}{2}, \dots, \frac{1}{2})} & \text{in } L^2_{\text{disc}}(Sp_n) \\ E(\tau, \lambda) & \longrightarrow & \end{array}$$

Let $\sigma(\tau) = \text{FJ}(\mathcal{E}_{-1}^\tau)$ - irr. cusp. gen. ant. rep. of \tilde{Sp}_n . (G-R-S)

Thm (Lapid- σ): $(G, H) = (\mathrm{Sp}_m, \mathrm{Sp}_m \times \mathrm{Sp}_m)$

P_H is abs. conv. on \mathcal{E} and
is H -dist.

With Lapid we have further results on the distinguished spectrum. We hope to apply them with soundry

to show that the descent construction gives all the cuspidal generic spectrum of $\tilde{\mathrm{Sp}}_m$.

